A 50-MINUTE INTRODUCTION TO AFFINE ALGEBRAIC GROUPS

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ABSTRACT. In this talk, we will introduce the notion of affine algebraic groups, starting with basic definition and examples. The first half of the talk will focus on the Lie algebras associated with affine algebraic groups. In the second half of the talk, we will present several basic theorems, including Chevalley's theorems on closed and normal subgroups, and Borel's fixed point theorem. Using Chevalley's theorem, we will give a meaningful intepretation of quotients of affine algebraic groups. Based on Borel's fixed point theorem, we will give a quick and dirty proof of Lie-Kolchin theorem. Lecture notes will be distributed at the beginning of the talk.

CONTENTS

In this talk, k denotes an algebraically closed field. Starting from section 3, k will be assumed to be characteristic 0 to avoid separability issues.

1. BASIC DEFINITIONS, EXAMPLES AND PROPERTIES

We start with the notion of algebraic groups.

Definition 1.1 (Algebraic Groups). An algebraic group is an algebraic variety G with a distinguished point $e \in G$, and two morphisms of varieties

$$\mu : G \times G \to G,\ (g_1, g_2) \mapsto g_1 \cdot g_2;\ \iota : G \to G,\ g \mapsto g^{-1}$$

Date: November 17, 2017, Student Algebraic Geometry Seminar.

such that G is a group with identity e, group operation μ and inverse ι.

We are primarily interested in the case in which the underlying variety is affine (but not necessarily irreducible). Under this assumption we say G is an affine algebraic group. Starting from this point, we assume all algebraic groups are affine, unless the contrary is explicitly stated.

Example 1.2 ($\mathbf{G_a}$ and $\mathbf{G_m}$)**.** The additive group $\mathbf{G_a}$ is the affine line $\mathbb{A}^1$ with identity $e = 0$, group operation $\mu(a_1, a_2) = a_1 + a_2$, and inverse $\iota(a) = -a$.

The multiplicative group $\mathbf{G_m}$ is the principal open subset $\mathbb{A}^1 \backslash \{0\} \subseteq \mathbb{A}^1$ with identity $e = 1$, group operation $\mu(a_1, a_2) = a_1 a_2$, and inverse $\iota(a) = \frac{1}{a}$.

Example 1.3 ($\mathrm{GL}_n(k)$)**.** The general linear $\mathrm{GL}_n(k)$ is the principal open subset $D_{\mathrm{Mat}_{n\times n}(k)}(\det)$ with identity being the $n \times n$ identity matrix, group operation being matrix multiplcation, and inverse being matrix inversion.

Example 1.4. Every closed subgroup of an algebraic group is an algebraic group.

Example 1.5 (Classical Matrix Groups)**.** There are four families of classical matrix groups, denoted A_l, B_l, C_l, D_l ($l \in \mathbb{Z}_{\geq 1}$) respectively, which are all closed subgroups of general linear groups. They are central to the theory of affine algebraic groups.

(1) A_l: The special linear group $\mathrm{SL}_{l+1}(k)$ consists of matrices of determinant 1.

(2) B_l: For k with $\mathrm{char}(k) \neq 2$, the special orthogonal group $\mathrm{SO}_{2l+1}(k)$ consists of matrices $A \in \mathrm{SL}_{2l+1}(k)$ satisfying

$$A^t \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & J \\ 0 & J & 0 \end{pmatrix} A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & J \\ 0 & J & 0 \end{pmatrix},$$

where J denotes the $n \times n$ matrix with 1 on the anti-diagonal.

(3) C_l: The symplectic group $\mathrm{Sp}_{2l}(k)$ consists of matrices $A \in \mathrm{GL}_{2l}(k)$ satisfying

$$A^t \begin{pmatrix} 0 & J \\ -J & 0 \end{pmatrix} A = \begin{pmatrix} 0 & J \\ -J & 0 \end{pmatrix}.$$

(4) D_l: For k with $\mathrm{char}(k) \neq 2$, the special orthogonal group $\mathrm{SO}_{2l}(k)$ consists of matrices $A \in \mathrm{SL}_{2l}(k)$ satisfying

$$A^t \begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix} A = \begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix}.$$

Being an affine variety, an algebraic group admits an irreducible decomposition $G = X_1 \cup \cdots \cup X_r$ in which the irreducible components X_i are cosets of the identity component of G:

Proposition 1.6. *Let G be an algebraic group with identity element e.*

(1) *e is contained in exactly one irreducible component G° (known as the identity component) of G.*

(2) *G° is a closed normal subgroup of finite index of G.*
(3) *The cosets of G° are exactly the irreducible components as well as the connected components of G.*
(4) *If H is a closed subgroup of finite index in G, then $H \supseteq G^\circ$.*

Definition 1.7 (Morphisms of Algebraic Groups). A morphism $f : G_1 \to G_2$ of two algebraic groups is a morphism of the underlying varieties that is compatible with the group structure. When $G_2 = \mathrm{GL}_n(k)$, we say $f : G_1 \to G_2 = \mathrm{GL}_n(k)$ is rational representation of G_1.

After we introduce the Lie algebra $\mathfrak{g}$ of an algebraic group G, we will relate rational representations of G with representations of the $\mathfrak{g}$.

The following proposition records some properties of morphisms of algebraic groups:

Proposition 1.8. *Let $f : G_1 \to G_2$ be a morphism of algebraic groups.*

(1) $\ker f$ *is a closed subgroup of G_1. Hence* $\ker f$ *is an affine algebraic group.*
(2) $\mathrm{Im}\, f$ *is a closed subgroup of G_2. Hence* $\mathrm{Im}\, f$ *is an affine algebraic group.*
(3) $f(G_1^\circ) = f(G_1)^\circ$.

Definition 1.9 (Action of Algebraic Groups). Let G be an algebraic group, X be a variety. An algebraic group action of G on X is a group action such that the action map $G \times X \to X$ is a morphism of varieties. X is then said to be a G-space.

Understanding algebraic group action turns out to be very useful for understanding the structure of algebraic group. This will be evident after we see Borel's fixed point theorem and Lie-Kolchin theorem. A important tool is the following:

Proposition 1.10 (Closed Orbits Lemma). *Let G be an algebraic group acting on a nonempty variety X. Then closed orbit exists.*

One key fact about affine algebraic groups is that all of them are linear:

Theorem 1.11 (Linearization of Affine Algebraic Groups). *An affine algebraic group G is isomorphic to a closed subgroup of some* $\mathrm{GL}_n(k)$.

This is the reason for which people usually call affine algebraic groups as linear algebraic groups.

2. Lie Algebras of Algebraic Groups

Before introducing Lie algebras, we shall brief review the concepts of tangent spaces, differentials and dimensions of varieties.

Definition 2.1 (Point Derivations and Tangent Space). *Let X be an irreducible variety. Let $x \in X$ be a point with local ring $\mathcal{O}_{X,x}$. A point derivation δ at x is a map $\delta : \mathcal{O}_{X,x} \to k$ such that:*

(1) δ *is* $k-$*linear.*
(2) $\delta(f\cdot g)=\delta(f)\cdot g(x)+f(x)\cdot\delta(g)$.

The tangent space T_xX *is defined to be the set of all point derivation* δ *at* x *of* $\mathcal{O}_{X,x}$.

Definition 2.2 (Differential of a Morphism). Let $f:X\to Y$ be a morphism of irreducible varieties. The differential of f at $x\in X$ is

$$d_xf:T_xX\to T_{f(y)}Y,\ \delta\mapsto d_xf(\delta),$$

where $d_xf(\delta)=\delta\circ f^{\#}:\mathcal{O}_{Y,y}\xrightarrow{f^{\#}}\mathcal{O}_{X,x}\xrightarrow{\delta}k$.

Easy computations show that d_xf is a $k-$linear map between two $k-$vector spaces T_xX and $T_{f(y)}Y$; and $d_xf(\delta)$ is indeed a point derivation at $f(y)$. Also, the chain rule follows easily from the definition.

Lemma 2.3 (Chain Rule). *Suppose* $f:X\to Y$, $g:Y\to Z$ *are morphisms. Let* $x\in X$, $y=f(x)\in Y$, $z=g(f(x))\in Z$. *Then* $d_x(g\circ f)=d_yg\circ d_xf:T_xX\to T_zZ$.

Example 2.4. (1) Suppose X and Y are affine varieties embedded in $\mathbb{A}^n$, $\mathbb{A}^m$ respectively. Say $f=(f_1,\cdots,f_m):X\to Y$ with $f_i\in k[x_1,\cdots,x_n]$, then $d_xf:T_x\mathbb{A}^n\to T_{f(x)}\mathbb{A}^m$ is given by the Jacobian matrix $d_x(f)=\left(\frac{\partial f_j}{x_i}\right)_{1\leq i\leq i,1\leq j\leq m}$.

(2) Consider the canonical projection $\pi:\mathbb{A}^n_k\backslash\{0\}\to\mathbb{P}^{n-1}_k$. Recall $\mathbb{P}^{n-1}_k$ has a covering by affine open sets $\mathbb{P}^{n-1}_k=\cup_{i=1}^nU_i$:

$$U_i=\{(a_1:\cdots:a_i\cdots:a_n):a_i\neq 0\}\overset{\cong}{\leftrightarrow}\mathbb{A}^{n-1}_k,\ (a_1:\cdots:a_n)\leftrightarrow(\frac{a_1}{a_i},\cdots,\frac{a_{i-1}}{a_i},\frac{a_{i+1}}{a_i},\cdots,\frac{a_n}{a_i}).$$

Let $v=(1,0,\cdots,0)\in\mathbb{A}^n_k\backslash\{0\}$, $\pi(v)=(1:0:\cdots:0)\in U_1$. Composition of morphisms allows us to compute $d_v\pi$ in affine coordinates:

$$\mathbb{A}^n_k\backslash\{(0,x_2,\cdots,x_n):x_i\in k\}\xrightarrow{\pi}U_1\leftrightarrow\mathbb{A}^k_{n-1},\ (x_1,x_2\cdots,x_n)\to(x_1:x_2:\cdots:x_n)\leftrightarrow(\frac{x_2}{x_1},\cdots,\frac{x_n}{x_1})$$

By (1), $d_v\pi=\begin{pmatrix}0&1&0&\cdots&0\\0&0&1&\cdots&0\\\vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&\cdots&1\end{pmatrix}$. Its kernel is precisely the subspace $k\cdot v$. This holds for all $v\in\mathbb{A}^n\backslash\{0\}$.

The computation in (2) will be useful when we construct quotient G/H of algebraic groups. A more directly related example of differentials is the following:

Example 2.5. We determine the differential of μ and ι for an algebraic group G.

(1) Consider $d_{(e,e)}G\times G\cong d_eG\oplus d_eG\to d_eG$. A point derivation $\delta\in T_{(e,e)}G\times$ can be identified with $(\delta_1,\delta_2)\in T_eG\oplus T_eG$.

Let $f \in \mathcal{O}_{G,e}$, $\mu^{\#}(f) = \sum_i h_i \otimes g_i$, $h_i, g_i \in \mathcal{O}_{G,e}$, i.e., $f(x \cdot y) = \sum_i h_i(x) g_i(y)$. Setting $x = e$ or $y = e$, then $f(x) = \sum_i h_i(x) g_i(e)$, $f(y) = \sum_i h_i(e) g_i(y)$. So

$$\delta_1(f) + \delta_2(f) = \sum_i \delta_1(g_i) h_i(e) + \sum_i g_i(e) \delta_2(h_i).$$

On the other hand,

$$d_{(e,e)}\mu(\delta)(f) = \delta(\mu^{\#} f) = (\delta_1, \delta_2)(\sum_{1 \le i \le j} h_i \otimes g_i) = \sum_{1 \le i \le j} \delta_1(g_i) h_i(e) + g_i(e) \delta_2(h_i).$$

We conclude that $d_{(e,e)}\mu(\delta) = d_{(e,e)}\mu(\delta_1, \delta_2) = \delta_1 + \delta_2$.

(2) Consider $d_e \iota : T_e G \to T_e G$. Note the composition

$$G \overset{(\mathrm{Id}, \iota)}{\to} G \times G \overset{\mu}{\to} G,$$

is a constant map on G, thus the differential is trivial. By chain rule,

$$0 = d_{(e,e)}\mu \circ d_e(\mathrm{Id}, \iota) = d_{(e,e)}\mu(d_e \mathrm{Id}(\delta), d_e\iota(\delta)) \overset{(1)}{=} \delta + d_e\iota(\delta) \Rightarrow d_e\iota(\delta) = -\delta.$$

Recall a point x on a variety X is said to be smooth if $\dim T_x X = \dim X$. Smooth points alway exist for irreducible varieties. This is the content of the next theorem:

Theorem 2.6. *Let X be an irreducible affine variety.*

(1) $\dim T_x X \geq \dim X$ *for all* $x \in X$.

(2) *The set of smooth points forms an open (hence dense) subset of* X.

For an algebraic group G, the left translation by $g \in G$, $G \to G$, $x \mapsto \mu(g, x)$, is an isomorphism of the underlying variety. Its differential $T_x G \to T_{g \cdot x} G$ is an vector space isomorphism, thus $\dim T_x G$ is a constant for all $x \in G$. By above theorem, smooth points exist in G so every point is smooth:

Proposition 2.7. *All points on an algebraic group G are smooth.*

Morphisms of algebraic groups satisfy the "rank-nullity theorem":

Proposition 2.8. $\dim G_1 = \dim \ker f + \dim \mathrm{Im}\, f$ *for an algebraic group morphism* $f : G_1 \to G_2$.

A main theme of the theory of algebraic groups is to classify algebraic groups of various types. Classifying $1-$dimensional algebraic groups is not very challenging:

Theorem 2.9 (Classifying $1-$Dimensional Groups). *The only connected $1-$dimensional algebraic groups (up to isomorphism) are $\mathbf{G_a}$ and $\mathbf{G_m}$.*

We give some other examples for dimension of algebraic groups.

Example 2.10 ($\mathrm{GL}_n(k), \mathrm{SL}_n(k)$). Since $\mathrm{GL}_n(k)$ is a principal open subset, $\dim \mathrm{GL}_n(k) = n^2$. As $\mathrm{SL}_n(k)$ is cut off by one equation, $\dim \mathrm{SL}_{l+1}(k) = (l+1)^2 - 1$.

The dimension of other three families of classical matrix groups is not possible to read off directly from their defining equations. We need Lie algebras to determine their dimension.

Definition 2.11 (Lie Algebras). *A Lie algebra is a $k-$vector space $\mathfrak{g}$ equipped with Lie bracket $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ such that*

(1) *(Bilinearity) $[\cdot,\cdot]$ is linear in each variable.*
(2) *(Antisymmetry) For all $x \in \mathfrak{g}$, $[x, x] = 0$.*
(3) *(Jacobi Identity) For $x, y, z \in \mathfrak{g}$, $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$.*

Example 2.12 (Matrix Algebra). The $k-$vector space of $n \times n$ matrices with entries from k becomes a Lie algebra under the commutator bracket $[A, B] = AB - BA$. It is denoted $\mathfrak{gl}_n(k) := (\mathrm{Mat}_n(k), [\cdot,\cdot])$. Subalgebras of $\mathfrak{gl}_n(k)$ is called linear Lie algebra.

Definition 2.13 (Lie Algebra Homomorphisms). *Let $(\mathfrak{g},\ [\cdot,\cdot]_{\mathfrak{g}})$ and $(\mathfrak{h},\ [\cdot,\cdot]_{\mathfrak{h}})$ be two Lie algebras. A Lie algebra homomorphism $\phi : \mathfrak{g} \to \mathfrak{h}$ is a $k-$linear map such that $[\phi x, \phi y]_{\mathfrak{h}} = \phi[x, y]_{\mathfrak{g}}$ for all $x, y \in \mathfrak{g}$. When $\mathfrak{h} = \mathfrak{gl}_n(k)$, we say $\phi : \mathfrak{g} \to \mathfrak{h} = \mathfrak{gl}_n(k)$ is a representation of $\mathfrak{g}$.*

Recall every affine algebraic group is linear. Similarly we have Ado's theorem for Lie algebras:

Theorem 2.14 (Ado). *Every finite dimensional Lie algebra is isomorphic to a Lie subalgebra of some general linear algebra $\mathfrak{gl}_n(k)$.*

Previously we have seen the notion of point derivations. Now a derivation of an algebraic group G is a map $\delta : k[G] \to k[G]$ such that

$$\delta(f + h) = \delta(f) + \delta(h),\ \delta(f \cdot h) = f \cdot \delta(h) + \delta(f) \cdot h.$$

The $k-$vector space of all derivations of $k[G]$ is denoted $\mathrm{Der}(k[G])$.

Definition 2.15 (Left Invariant Derivations). *Let $g \in G$. The map*

$$\lambda_g : k[G] \to k[G],\ f \mapsto (g' \mapsto f(g^{-1} \cdot g'))$$

is called left translation of functions by g. A derivation $\delta : k[G] \to k[G]$ is left invariant if $\delta \circ \lambda_g = \lambda_g \circ \delta$ for all $g \in G$. The $k-$vector subspace of all left invariant derivations of $k[G]$ is denoted $\mathscr{L}(G)$.

Definition 2.16 (Lie Algebra of G). *For $\delta_1, \delta_2 \in \mathscr{L}(G)$, define*

$$[\cdot,\cdot] : \mathscr{L}(G) \times \mathscr{L}(G) \to \mathscr{L}(G),\ [\delta_1, \delta_2] : \delta_1 \circ \delta_2 - \delta_2 \circ \delta_1.$$

$[\cdot,\cdot] : \mathscr{L}(G) \times \mathscr{L}(G) \to \mathscr{L}(G)$ is well-defined and makes $\mathscr{L}(G)$ into a Lie algebra.

One should think in the way that derivations are global and yield points derivations at each point. More formally, for $\delta \in \mathscr{L}(G)$, denote $\theta(\delta)$ to be such that

$$\theta(\delta) : k[G] \to k, \ f \mapsto (\delta(f))(e),$$

where e denotes the identity element of G. Such $\theta(\delta)$ will be a point derivation at e, i.e, $\theta(\delta)$ extends to a map

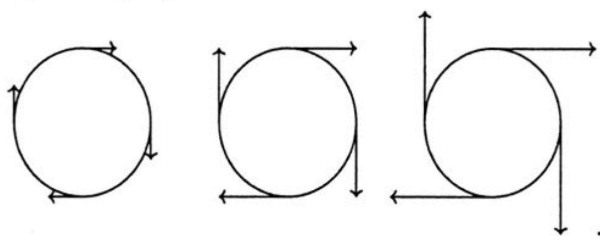
$$\theta(\delta) : \mathcal{O}_{G,e} \to k$$

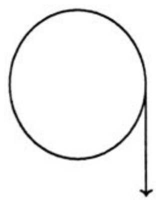
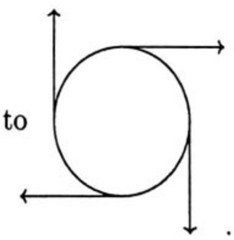
such that $\theta(\delta)$ is $k-$linear and it satisfies product rule:

$$\theta(\delta)(f \cdot h) = \theta(\delta)(f) \cdot h(e) + f(e) \cdot \theta(\delta)(h).$$

Hence we have an assignment $\theta : \mathscr{L}(G) \to T_e(G)$.

The key fact is that θ is actually a $k-$vector space isomorphism. A point derivation at e may be pushed forward to all of $g \in G$ if we insist it being preserved under left translation. This idea of "push forward" a point derivation to a "global" derivation can be summarized in the following picture, though this is really a picture of the Lie group S^1, not a picture of affine algebraic group:

Left invariant derivations:  .

A point derivation  pushes forward to  .

Theorem 2.17 (Tangent Space as Lie Algebra). *$\theta : \mathscr{L}(G) \to \mathfrak{g}$ is a vector space isomorphism.*

Proof Sketch. Construct an inverse $* : \mathfrak{g} \to \mathscr{L}(G)$ of θ, known as convolution:

$$(*\eta(f))(g) = \eta(\lambda_{g^{-1}}(f))$$

for $f \in k[G]$ and $g \in G$. One needs to show the following:

(1) The image of $*\eta$ does lie in $k[G]$.
(2) $*\eta : k[G] \to k[G]$ is a derivation on $k[G]$.
(3) $*\eta : k[G] \to k[G]$ is left invariant.
(4) $* : \mathfrak{g} \to \mathscr{L}(G)$ and θ are mutual inverse of each other.

The details will be omitted. □

The Lie algebra of $\mathrm{GL}_n(k)$ is $\mathfrak{gl}_n(k)$:

Theorem 2.18 (Lie Algebra of the General Linear Group). *The composition*

$$\mathscr{L}(\mathrm{GL}_n(k)) \overset{2.17}{\leftrightarrow} T_{I_n}\,\mathrm{GL}_n(k) \overset{\heartsuit}{\leftrightarrow} \mathfrak{gl}_n(k)$$

is a Lie algebra isomorphism.

Proof Sketch. The map $\heartsuit$ is $\sum_{i,j=1}^n A_{i,j}\frac{\partial}{\partial X_{i,j}}|_{I_n} \leftrightarrow (A_{i,j}) = \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \cdots & \cdots & \cdots \\ A_{n,1} & \cdots & A_{n,n} \end{pmatrix}$. □

We return to the question what is the dimension of classical matrix groups:

Example 2.19 (Lie Algebras of Classical Matrix Groups). In Lie algebra, there are also four families A_l, B_l, C_l, D_l of classical algebras, which are subalgebras of $\mathfrak{gl}_n(k)$. They are exactly the Lie algebras of corresponding classical matrix groups.

(1) A_l: Let $\mathfrak{sl}_{l+1}(k)$ be the space of all trace zero matrices $x \in \mathrm{Mat}_{l+1}(k)$, known as special linear algebra. It is the Lie algebra of $\mathrm{SL}_{l+1}(k)$.

(2) B_l: Let $\mathrm{char}\, k \neq 2$. Let $f : k^{(2l+1)} \times k^{(2l+1)} \to k$ be the bilinear form whose matrix is $\left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & \mathrm{Id}_l \\ 0 & \mathrm{Id}_l & 0 \end{smallmatrix}\right)$. Let $\mathfrak{o}_{2l}(k)$ be the space of matices $x \in \mathrm{Mat}_{2l+1}(k)$ such that

$$f(x(v), w) = -f(v, x(w)) \ \forall\ v, w \in k^{2l+1},$$

known as orthogonal algebra. If is the Lie algebra of $\mathrm{SO}_{2l+1}(k)$.

(3) C_l: Let $f : k^{2l} \times k^{2l} \to \mathbb{F}$ be the bilinear form whose matrix is $\left(\begin{smallmatrix} 0 & \mathrm{Id}_l \\ -\mathrm{Id}_l & 0 \end{smallmatrix}\right)$. Let $\mathfrak{sp}_{2l}(k)$ to be the space of matrices $x \in \mathrm{Mat}_{2l}(k)$ such that

$$f(x(v), w) = -f(v, x(w)) \ \forall\ v, w \in k^{2l},$$

known as sympletic algebra. It is the Lie algebra of $\mathrm{Sp}_{2l}(k)$.

(4) D_l: Let $\mathrm{char}(\mathbb{F}) \neq 2$. Let $f : k^{2l} \times k^{2l} \to k$ be the bilinear form whose matrix is $\left(\begin{smallmatrix} 0 & \mathrm{Id}_l \\ \mathrm{Id}_l & 0 \end{smallmatrix}\right)$. Let $\mathfrak{o}_{2l}(V)$ to be the space of matrices $x \in \mathrm{Mat}_{2l}(V)$ such that

$$f(x(v), w) = -f(v, x(w)) \ \forall\ v, w \in V,$$

also called orthogonal algebra as in (2). It is the Lie algebra of $\mathrm{SO}_{2l}(k)$.

To show these are the Lie algebras of the matrix groups, one can use "dual numbers". Now computing the dimension of classical matrix groups downgrades to computing the dimension of these classical algebras, which is a easier linear algebra problem.

One consequence of theorem 2.17 is that we may start to call $\mathfrak{g} = T_e G$ as the Lie algebra of the affine algebraic group G, by pulling the structure of Lie algebra on $\mathscr{L}(G)$ back to $\mathfrak{g}$. Using this convention, we now answer the question how morphism of algebraic groups is related to morphism of Lie algebras:

Theorem 2.20 (Induced Lie Algebra Homomorphism). *Let G and H be affine algebraic groups with Lie algebra $\mathfrak{g} = T_eG$ and $\mathfrak{h} = T_eH$ respectively. Let $\phi : G \to H$ be a affine algebraic group homomorphism. Then the differential*

$$d_e\phi : \mathfrak{g} \to \mathfrak{h},\ \eta \mapsto d_e\phi(\eta),\ d_e\phi(\eta)(f) = \eta(f \circ \phi),\ f \in k[G']$$

is a Lie algebra homomorphism.

In particular, a rational representation $\varphi : G \to \mathrm{GL}_n(k)$ of an affine algebraic group gives rise to a representation of Lie algebra $d_e\varphi : \mathfrak{g} \to \mathfrak{gl}_n(k)$.

3. Quotients of Algebraic Groups

Starting from now on, we assume $\operatorname{char} k = 0$ to avoid separability problems.

In group theory, one constructs coset space and quotient group. When N is a normal subgroup of G, then the quotient G/N is a group and $\pi : G \to G/N$ is a group homomorphism. $(G/N, \pi)$ satisfies universal mapping property that if $\phi : G \to G'$ is a group homomorphism such that $N \subseteq \ker(\phi)$, then there exists a unique group homomorphism $\tilde{\phi} : G/N \to G'$ such that $\phi = \tilde{\phi} \circ \pi$.

The goal of this section is to make sense of G/H for a closed subgroup H of an affine algebraic group G, and describe its universal property. In order to do so, one needs the help of two Chevalley's theorems.

Theorem 3.1 (Chevalley 1). *Let G be an algebraic group, H be a closed subgroup of G. Then there is a rational representation $\varphi : G \to \mathrm{GL}(V)$ and an $1-$dimensional subspace $L \subseteq V$ such that*

$$H = \{x \in G : \varphi(x)L = L\},\ \mathfrak{h} = \{\delta \in \mathfrak{g} : d\varphi(\delta)L \subseteq L\}.$$

Definition 3.2 (Homogeneous $G-$Space). *Let G be an algebraic group and X be a $G-$space. X is said to a homogeneous provided that the $G-$action is transtitive.*

Definition 3.3 ($G-$Equivariant Morphism). *A morphism $\phi : X \to Y$ between two $G-$spaces is said to be $G-$equivariant provided that following diagram commutes:*

$$\begin{tikzcd} G \times X \arrow[r] \arrow[d, "\mathrm{Id} \times \phi"'] & X \arrow[d, "\phi"] \\ G \times Y \arrow[r] & Y \end{tikzcd}.$$

In concrete formula, $\phi(g \cdot x) = g \cdot \phi(x)$.

Theorem 3.4. *Let H be a closed subgroup of an algebraic group G.*

(1) *The coset space G/H has a structure of quasi-projective variety, and $\pi : G \to G/H$ is a morphism of varieties.*

(2) *Let X be a homogeneous $G-$space, $\phi : G \to X$ be a $G-$equivariant morphism such that $\phi(h) = \phi(1)$ for all $h \in H$. Then there exists a unique $G-$equivariant morphism $p\tilde{h}i : G/H \to X$ such that $\phi = \tilde{\phi} \circ \pi$.*

Proof Sketch. By 3.1, there is a rational representation $\varphi : G \to \mathrm{GL}(V)$ and an $1-$dimensional subspace $L \subseteq V$ such that

$$H = \{x \in G : \varphi(x)L = L\},\ \mathfrak{h} = \{\delta \in \mathfrak{g} : d\varphi(\delta)L \subseteq L\}.$$

We may identify H to be the stabilizer the point $[L] \in \mathbb{P}(V)$. Orbit-Stabilizer theorem gives a bijection $G/H \leftrightarrow G\cdot[L]$. The orbit $G\cdot[L]$ is in fact quasi-projective. Use this bijection to endow G/H with a structure of quasi-projective variety and a sheaf of function. □

The previous Chevalley's theorem assumes H only to be closed. The next Chevalley's theorem described the situation for normal closed subgroups.

Theorem 3.5 (Chevalley 2). *Let G be an algebraic group, N be a closed normal subgroup of G. Then there is a rational representation $\varphi : G \to \mathrm{GL}(W)$ such that*

$$N = \ker\varphi,\ \mathfrak{n} = \ker d\varphi.$$

This version of Chevalley's theorem implies that quotient of an affine algebraic group by a normal closed subgroup is again an affine algebraic group.

Theorem 3.6. *Let N be a normal closed subgroup of an algebraic group G. Then the coset space G/N has the structure of an affine algebraic group.*

Proof Sketch. By 3.5, there is a morphism of algebraic group $\varphi : G \to \mathrm{GL}_n(k)$ such that $\ker\varphi = N$. As abstract groups, $G/N \cong \varphi(G) \subseteq \mathrm{GL}_n(k)$. In fact this is an isomorphism of varieties. □

4. Borel's Fixed Point Theorem and Lie-Kolchin Theorem

Historically, the notion of solvable groups emerged when people tackled the problem whether quintic polynomials are solvable in radicals. In this section, we will show all connected solvable affine algebraic groups are upper triangular. Let's recall the definition of solvable groups first:

Definition 4.1 (Solvable Groups). *The derived series of an abstract group G is*

$$D^0(G) \supseteq D^1(G) \supseteq D^2(G) \supseteq \cdots$$

where $D^0(G) = G$, $D^{i+1}(G) = [D^i(G), D^i(G)]$. G is said to be solvable provided its derived series ends with $D^n(G) = e$ for some n.

Here is an easy fact about solvable groups:

Lemma 4.2. *Subgroups and homomorphic images of solvable groups are solvable.*

Remark 4.3. One built-in feature of the definition of solvable group is that it allows induction on dimension of connected solvable affine algebraic groups. The reason is that if G is connected solvable affine algebraic group, then (G, G) is closed, connected and solvable with $\dim(G, G) < \dim G$. We will use this feature in the proof of Borel's fixed point theorem.

Example 4.4 (Upper Triangular Matrices). $\mathbf{T}_n(k)$ denotes the set of invertible upper triangular $n \times n$ matrices, which constitutes a closed subgroup of $\mathrm{GL}_n(k)$. $\mathbf{T}_n(k)$ is solvable, but the argument for showing this is not extremely enjoyable. We downgrade it to an exercise.

We are now ready for Borel's fixed point theorem:

Theorem 4.5. *If G is a connected solvable affine algebraic group acting on a nonempty projective variety, then G has a fixed point on X.*

Proof. As promised before, we will use induction on $\dim G$. If $\dim G = 0$, then $G = \{e\}$ fixes every point of X, proving the base case. In the induction step, we use induction hypothesis to (G, G) so that (G, G) has fixed points on X. Let Y be an irreducible subvariety of X which (G, G) acts trivially. Since (G, G) is normal in G, Y is invariant under $G-$action, then $H := G/(G, G)$ acts on Y. By closed orbit lemma 1.10, there exists a closed orbit $Z \subseteq Y$ of H. Z is a projective variety, and is isomorphic to H/H_x where H_x is the stabilizer of some $x \in Z$. Since H is abelian, H_x is normal in H, so by 3.6, H/H_x is an affine algebraic group. It follows that $Z \cong H/H_x$ is irreducible, affine and projective, thus must be trivial. We conclude Z consists of one point, and this point is fixed by G. □

Having Borel's fixed point theorem in hand, we will easily obtain Lie-Kolchin theorem which asserts every connected solvable affine algebraic groups are upper triangular. The last missing of puzzle is flag variety before we prove Lie-Kolchin theorem.

Definition 4.6 (Flag Variety). *Let V be a $d-$dimensional $k-$vector space. The flag variety $\mathfrak{F}(V)$ consists of full flags*

$$0 \subseteq V_1 \subseteq \cdots \subseteq V_{d_1} \subseteq V$$

where $\dim V_j = j$. $\mathfrak{F}(V)$ has a structure of projective variety.

Theorem 4.7. *Let G be a connected solvable affine algebraic group. Let $\varphi : G \to \mathrm{GL}(V)$ be a rational representation. There exists a basis of V such that the images in $\varphi(G)$ are all upper triangular, i.e., $\varphi(G) \subseteq \mathbf{T}_n$.*

Proof. $\varphi(G)$ is a closed connected solvable subgroup of $\mathrm{GL}(V)$. $\varphi(G)$ acts on the flag varieties $\mathfrak{F}(V)$. Since $\mathfrak{F}(V)$ is complete, by Borel's fixed point theorem 4.5, $\varphi(G)$ fixes a flag

$$0 = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_n = V.$$

With respect to a suitable basis, $\varphi(G)$ is upper triangular. □

5. Acknowledgement

I would like to thank Matt Stevenson for giving me this chance of presenting at the Student Algebraic Geometry Seminar; thank Professor Tasho Kaletha for his mentoring; thank Daniel Minahan and Sam Freedman for discussing affine algebraic groups during the summer.

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